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The two-dimensional tricritical Ising model in an external magnetic field

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Abstract. The two-dimensional tricritical Ising model with an order parameter perturbation is studied using two different lattice models. The mass spectrum is found to be the same as in the 2D Ising model with a magnetic field. Results for the scaling functions are in agreement with universality and for the lowest mass a relationship with the scaling function of the Ising model is observed. The numerical results do not support a recent proposal of Dotsenko on the dynamical generation of extra terms in the perturbative calculation of scaling functions.

1. Introduction

The hypothesis of conformal invariance has led to a considerable improvement in the description and understanding of two-dimensional critical phenomena (for a review, see [1]). Results include the exact determination of the central charge, critical exponents and correlation functions for numerous physical systems. However, a complete understanding of a critical point should also include the universal properties of the scaling region, where the correlation length is finite but much larger than any microscopic scale. In particular, one would like to calculate universal combinations of the bulk scaling amplitudes. This is particularly interesting for the distinction of universality classes, since critical amplitudes may vary over a much wider range of values than critical exponents (for a review, see [2]). For example, it is possible, using Zamolodchikov's [3] c-theorem, to express the universal scaling amplitude ratio $f^{(s)}\xi^2$ in terms of the central charge and critical exponents [4], where $f^{(s)}$ is the singular part of the free energy density and ξ is defined by the second moment of the two-point correlation function of the perturbing relevant field. As another example, we mention the calculation of the perturbative corrections due to a relevant operator for the central charge of a system flowing between two renormalization group fixed points [5].

Recently, it was suggested by Zamolodchikov [6], that if the critical point Hamiltonian is perturbed by a suitably chosen relevant scaling field, the off-critical system may possess non-trivial integrals of motion and might even be integrable. Integrals of motion of the form $:T^s(z):$, where $T(z)$ is the energy momentum tensor and s an integer, are found for the perturbing fields φ_{12}, φ_{21} and φ_{13} where the indices are the usual Kac labels. By a bootstrap approach, the complete factorized S-matrix can then be obtained. The ratios of the masses of the particles contained in the massive particle field theory correspond to the ratios of several correlation lengths (e.g. spin–spin or

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energy-energy) in a statistical model. The class of models which can be treated by this method includes the 2D Ising model in a magnetic field and has led in this case to the prediction [6, 7] for the masses m_i:

$$\begin{aligned} \frac{m_2}{m_1} &= 2\cos\frac{\pi}{5} = 1.618\ldots \\ \frac{m_3}{m_1} &= 2\cos\frac{\pi}{30} = 1.989\ldots. \end{aligned} \tag{1.1}$$

There are further stable states above the continuum starting at $2m_1$.

The above conjecture for the Ising model was checked and confirmed by Hamiltonian spectrum calculations [8]. This is a non-trivial check that the assumptions involved in (1.1) appear at least to some extent to be satisfied in statistical systems. This check is further confirmed by correlation function computations [9] and the spectrum calculations were also repeated [10]. Similar predictions exist for example for the tricritical Ising model with thermal (φ_{12}) or vacancy concentration (φ_{13}) perturbations [11, 12] and were checked using the Blume-Capel model [13]. (Non-minimal models were studied as well, see for example the Yang-Lee edge singularity [14] or the Ashkin-Teller model with thermal perturbations [15].) The exact S-matrices have been obtained for these types of massive field theories and are shown to have the same Feynman graph expansion as extended Toda field theories [11, 12, 16-22]. (For example, the Ising model with a magnetic field corresponds to the E_8 Toda field theory.) Dispersion relations were also studied for the Ising model perturbed with a magnetic field ($\varphi_{12}=\varphi_{22}$) [8] and for the tricritical Ising model perturbed with φ_{12} and φ_{13} [13] and found to be consistent with those of a free massive particle as it is expected from the approach leading to (1.1) [6, 7].

Interest has mainly concentrated on the perturbing operators φ_{12}, φ_{21} and φ_{13} since these are the only ones for which non-trivial integrals of motion are known to exist. In particular, the effects of perturbing with φ_{13}, which is the least relevant operator for the conformal minimal models, has received considerable attention (e.g. [11, 12, 17, 23] and references therein). Here we want to study the other extreme case, that is perturbations with the operator φ_{22}, which is the *most relevant operator* in minimal models and corresponds to order parameter perturbations. However, the perturbative approach is plagued by IR divergencies which for the perturbing field φ_{22} already show up in first order. We shall therefore use a non-perturbative approach.

Here we shall consider the simplest non-trivial example, namely the tricritical Ising model (in the usual Ising model, one has $\varphi_{12}=\varphi_{22}$). Examples of tricritical behaviour may occur in metamagnets or multicomponent fluid mixtures. For a detailed review on tricritical behaviour see [24]. We shall study two different lattice realizations of this universality class.

1. The RSOS model [25, 26] using the transfer matrix. The transition between the regimes III and IV [25] shows (multi-)critical behaviour. A convenient formulation relates these models to Dynkin diagrams of simple Lie algebras, where the tricritical Ising model corresponds to the algebra A_4 [27].

2. The Hamiltonian limit version of an Ising metamagnet, described by the quantum Hamiltonian

$$\begin{aligned} H = -\sum_{n=1}^{N} [(t\sigma^z(n) - \sigma^x(n)\sigma^x(n+1) + \sigma^x(n)\sigma^x(n+2) + h\sigma^x(n)) \\ + h_s(-1)^n\sigma^x(n)] \end{aligned} \tag{1.2}$$

where σ^x and σ^z are Pauli matrices and periodic boundary conditions are understood. The variable t acts like a temperature, while h parametrizes the competition between the ferromagnetic and the antiferromagnetic ground states and h_s is the symmetry-breaking staggered field coupled to the order parameter. The quantum Hamiltonian H can be obtained in the usual way from the transfer matrix $T = \exp(-\tau H)$ of an isotropic metamagnet [28–30] by taking the extreme anisotropic limit along with $\tau \to 0$. This correspondence is well established and is reviewed in [31].

Using the RSOS model has the advantage that the location of the tricritical point is known exactly. Explicit expressions exist for the perturbing operators φ_{nn} of the Kac table [27]. On the other hand, using the Ising metamagnet offers more intuition on the correspondence with real systems and also allows us to consider perturbations which cannot be written in terms of the 'diagonal' operators φ_{nn}, but has the disadvantage that the tricritical point (t_t, h_t) has to be found. In any case, using two realizations of the same universality class offers the possibility to check explicitly for the universality of the scaling functions and we shall use this below.

The paper is organized as follows. In section 2 the finite-size scaling calculation of the desired scaling amplitudes is described. We shall find the mass ratios to be the same as for the 2D Ising model in a magnetic field. Scaling functions are obtained and checked for universality and relationships between the non-universal normalization constants appearing in the scaling form are discussed. We shall also observe a relation between the scaling functions of the spin–spin correlation length of the Ising and the tricritical Ising model. Section 3 deals with perturbative ideas for the calculation of integrals of motion and scaling functions. In particular, we examine a recent proposal of Dotsenko [32] which attempts to include input from the operator product expansion and claims that the perturbation series might also contain additional, dynamically generated, contributions. A comparison with our numerical results, however, does not appear to support this idea. Conclusions are given in section 4.

2. Finite-size scaling calculation

Finite-size scaling techniques are common to calculate critical points and critical exponents of (multi-) critical points and become particularly powerful in the Hamiltonian limit since the sparsity of the matrices to be diagonalized is increased as compared to the usual transfer matrix (for a review and the combination with conformal invariance, see [31]).

Consider the phase diagram of the Ising metamagnet, as shown in figure 1. The determination of the tricritical point proceeds along the standard finite-size scaling method of calculating the two lowest gaps $m_1(N)$, $m_2(N)$ for various numbers of sites N and extrapolating towards $N \to \infty$ the solutions of the equations

$$\begin{aligned} Nm_1(t, h, N) &= (N+2)m_1(t, h, N+2) \\ Nm_2(t, h, N) &= (N+2)m_2(t, h, N+2) \end{aligned} \tag{2.1}$$

as described in the literature (e.g. [28–30, 33]). We find for the location of the tricritical point

$$t_t = 1.0940(9) \qquad h_t = 1.8794(15). \tag{2.2}$$

At the tricritical point, the spectrum of H can be decomposed into the unitary irreducible representations of the Virasoro algebra of central charge $c = \frac{7}{10}$. The applications of conformal invariance to the Ising metamagnet (1.2) requires the correct

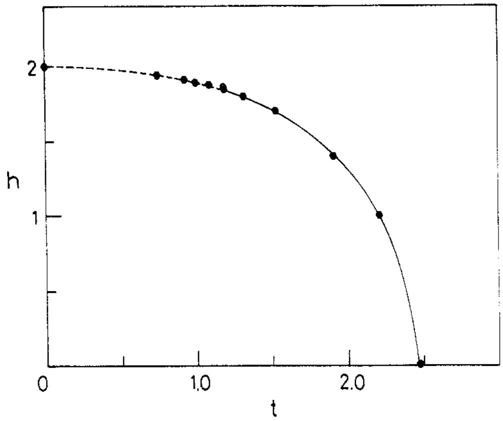

Figure 1. Phase diagram of the Ising metamagnet (1.2). The full line is in the 2D Ising universality class, while the broken line is a first-order transition. The tricritical point separating these lines is marked by a circle.

normalization of the Hamiltonian: $H \to \zeta^{-1}H$, where ζ can be fixed by standard arguments [34]. The best convergence is obtained, in our case, from the inverse energy-energy correlation length m_2 and we find

$$\zeta = 4.3 \pm 0.1 \tag{2.3}$$

for the Ising metamagnet, while the normalization problem, which reflects a certain arbitrariness in taking the Hamiltonian limit [31], does not arise in the isotropic transfer matrix of the RSOS model. The operator content of the tricritical Ising model can be obtained using the Hamiltonian limit of the Blume-Capel model [13, 33, 35]. There are four relevant scaling fields which correspond to the fields arising in the scaling renormalization group description of a tricritical point [24] as follows:

φ_{22}	$x = \frac{3}{40}$	order parameter
φ_{12}	$x = \frac{1}{5}$	temperature
φ_{21}	$x = \frac{7}{8}$	'cubic' field
φ_{13}	$x = \frac{6}{5}$	non-ordering field

where x is the scaling dimension. (More precisely, the 'true' temperature couples to a linear combination of φ_{12} and φ_{13}, see [13, 33].)

Now, consider the conformally invariant theory, given by the critical point Hamiltonian H_c, perturbed by some relevant scaling field φ

$$H = H_c + \lambda \int \varphi(z, \bar{z})\, d^2z. \tag{2.4}$$

Consider $\varphi = \varphi_{22}$ as a perturbing field. Then the coupling λ is the staggered field h_s for the Ising metamagnet (1.2). Define the scaling variable

$$\mu = h_s N^y \tag{2.5}$$

where $y = 2 - x = 77/40$. The masses m_i are calculated from the eigenvalues E_i of the Hamiltonian

$$m_i = \xi_i^{-1} = E_i - E_0 \tag{2.6}$$

where ξ_i is the correlation length of the ith scaling field where ξ_1 is the spin–spin correlation length, ξ_2 the energy–energy correlation length and so on. In figure 2, we show the first few mass ratios $r_i = m_{i+1}/m_1$ as a function of μ. Clearly, the r_i depend only on μ and not on h_s or N separately. The critical point corresponds to the limit $\mu \to 0$, that is $\xi_1 \gg N$. Conformal invariance predicts for $\mu = 0$ [33, 35]

$$r_1 = 2\tfrac{2}{3} \qquad r_2 = 11\tfrac{2}{3} \qquad r_3 = 16 \qquad r_4 = 27\tfrac{2}{3} \qquad \ldots \tag{2.7}$$

which are reproduced in figure 2. We now consider the other extreme case $\mu \to \infty$, that is $\xi_1 \ll N$. In this limit, a description of the system in terms of a massive field theory should be reasonable and the m_i become the masses of the particles in the field theory. Their ratios are related to the universal ratios of the scaling amplitudes B_i of the inverse correlation lengths $\xi_i^{-1} = B_i h_s^{1/y}$ via $m_i/m_1 = B_i/B_1$. Computationally, one has to get to this limit using finite-size calculations by a double extrapolation procedure. Two methods have been tried in practice.

1. Take first the limit $N \to \infty$ with h_s fixed and extrapolate afterwards back towards $h_s \to 0$. This technique was applied in the 2D Ising model [8].

2. Take first the finite-size scaling limit $N \to \infty$, $h_s \to 0$ and μ fixed and extrapolate afterwards for $\mu \to \infty$. This method was used in the Blume-Capel model [13].

We think that the first method is preferable, since it occurs quite often for larger values of μ that the finite-lattice data no longer form monotonous sequences (see the finite-size data in [13]) which renders the extrapolation towards $N \to \infty$ (μ fixed) quite difficult. We have not encountered this problem using the first method. Finally, it was also tried [10] to fit the finite-size data to the first few terms of the theoretical scaling expression where the finite-size scaling correction exponents are known from conformal invariance.

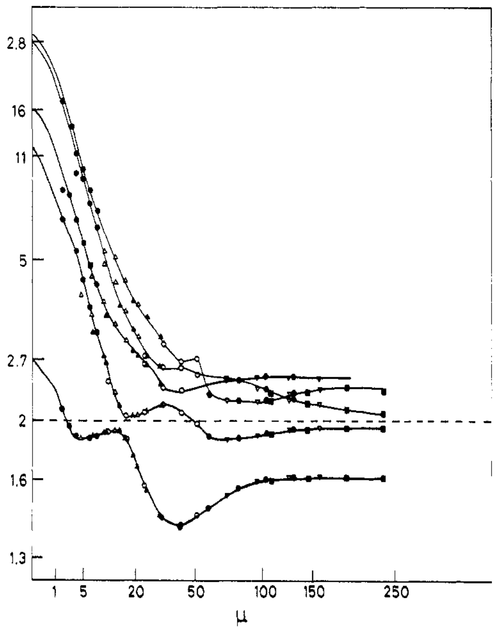

Figure 2. Mass ratios $r_i = m_{i+1}/m_1$ as a function of $\mu = h_s N^{77/40}$ as calculated for the Ising metamagnet (1.2). The symbols refer to the different values of h_s: ● 0.03, △ 0.08, ▲ 0.12, ○ 0.2, ⬤ 0.5, ▽ 0.6, ■ 0.9. The broken line marks the lower edge of the continuum starting at $r_i = 2$. The full curves are merely guides to the eye.

Although this method seems to work for the 2D Ising model, it is probably too dangerous in our case because of possible cross-over effects due to the nearby 2D Ising fixed point and we do not consider this technique any further.

To illustrate the convergence of our finite-size data, we display in table 1 two typical examples. Choosing the values of h_s needs some care since they have to be large enough to meet the condition $\xi \ll N$ but if h_s becomes too large, the masses eventually obtained may be affected by sizeable correction-to-scaling terms. The extrapolation was done with the BST algorithm (see [36, 37]) and we conclude from the results of the two models

$$m_2/m_1 = 1.62(1) \qquad m_3/m_1 = 1.98(2) \tag{2.8}$$

in the limit $\mu \to \infty$. Surprisingly, our numerical results agree with those obtained for the Ising model in a magnetic field (see (1.1)). We return to this point below.

Table 1. Finite-size data for the mass ratios $r_i = m_{i+1}/m_1$ for the Ising metamagnet calculated at $t = 1.094$ and $h = 1.8794$ for two values of the staggered field h_s.

	$h_s = 0.6$			$h_s = 0.9$		
N	r_1	r_2	r_3	r_1	r_2	r_3
8	1.404 72	1.702 61	2.120 89	1.406 15	1.749 39	1.880 40
10	1.418 26	1.906 02	2.034 58	1.534 03	1.823 58	2.160 11
12	1.525 01	1.847 87	2.192 37	1.591 90	1.881 80	2.167 20
14	1.592 41	1.879 01	2.172 56	1.602 55	1.924 17	2.155 27
16	1.613 12	1.925 02	2.200 31	1.604 33	1.938 33	2.106 51
18	1.618 18	1.946 81	2.165 31	1.604 65	1.943 90	2.074 02

Note that only the two lowest mass ratios in figure 2 can be followed continuously from their $\mu = 0$ value towards $\mu \to \infty$. All other ratios show level crossings where the corresponding states mix. Studies on the Ising and the Ashkin-Teller model indicate that mixing of states seems to occur generically between those levels which in the $\mu \to \infty$ limit are in the continuum part of the mass spectrum, while the discrete levels do not undergo mixing. This topic will be discussed in detail in a separate paper [38].

Further, the deep minima in the ratios r_i can be explained in the usual way [8]. Introduce the scaling functions G_i

$$m_i = h_s^{1/y} G_i(\mu) \tag{2.9}$$

$$G_i(\mu) = 2\pi x_i \mu^{-1/y} + H_i(\mu) \tag{2.10}$$

and the reduced scaling functions $H_i(\mu)$. The first term in (2.10) follows from conformal invariance. In figure 3, we plot $H_1(\mu)$ and $H_2(\mu)$ as obtained from the RSOS model and observe that they become small in the $\mu \to 0$ limit. As will be further discussed in section 3, for a symmetry-breaking field perturbation theory implies $H_i(\mu) \sim \mu^{2-1/y} + \dots$.

Since we have two realizations of the same universality class at our disposal, we can check for the universality of the finite-size scaling functions. We expect the finite-size scaling form [39], written down for the metamagnet in a staggered field

$$m_i = \xi_i^{-1} = N^{-1} S_i(C_2 h_s N^y) \tag{2.11}$$

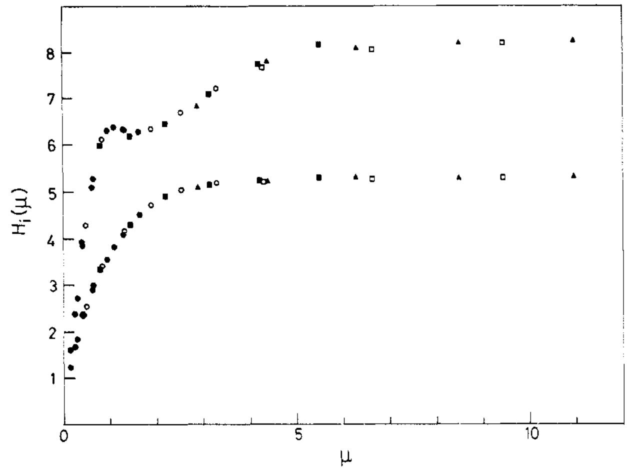

Figure 3. Reduced scaling functions $H_1(\mu)$ and $H_2(\mu)$ for the two lowest gaps in the A_4 RSOS model. The symbols correspond to the following values of the coupling λ: ● 0.02, ⬤ 0.03, ○ 0.06, ■ 0.10, △ 0.20, □ 0.30, and $\mu = \lambda N^{77/40}$.

where S_i is a universal function and all information about the specific system is contained in the metric constant C_2 (and in the corrections to (2.11)). Including the other relevant scaling fields would introduce additional metric constants, one for each additional scaling field [39]. In particular, there is no non-universal prefactor. Before applying (2.11) to the Hamiltonian limit, however, the Hamiltonian must be renormalized correctly.

In fact, the scaling functions for the metamagnet and the RSOS model are the same and one has only to rescale their arguments as shown in figure 4. We find from the lowest mass

$$\rho = C_2(\text{metamagnet})/C_2(\text{RSOS}) = 17.0 \pm 0.3. \tag{2.12}$$

Finite-size scaling can be used to relate ρ to the normalization of the quantum Hamiltonians, as we now show. Consider the critical point Hamiltonian H_c perturbed

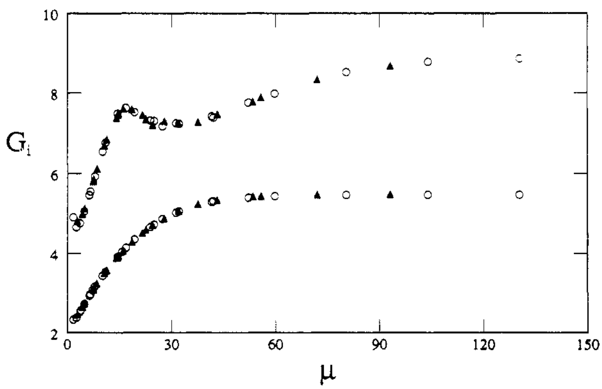

Figure 4. Comparison of the finite-size scaling functions G_i for the A_4 RSOS model (▲) and the Ising metamagnet (○). The scaling variable plotted corresponds to the Ising metamagnet and $\rho = 17$.

by $h\phi$, where ϕ is some relevant operator. We can assume that ϕ is normalized to unity, e.g. from its critical two-point correlation function. We take H_c to be correctly normalized so that its spectrum has integer spacing in terms of $2\pi/N$. In general, the system is not yet normalized and we have to work with the (quantum) Hamiltonian

$$H = \gamma(H_c + h\phi) = \gamma H_c + \tilde{h}\phi \tag{2.13}$$

where $\tilde{h} = \gamma h$. Consider two realizations of the same universality class characterized by triples $(\gamma, \tilde{h}, \tilde{C}_2)$ and $(\gamma', \tilde{h}', \tilde{C}_2')$. Since the lowest mass, for example, computed from both Hamiltonians corresponds to the same physical quantity, the correlation length ξ_1^{-1}, we can write

$$\xi_1^{-1} = \gamma N^{-1} S_1(\tilde{C}_2 \tilde{h} N^y) = \gamma' N^{-1} S_1(\tilde{C}_2' \tilde{h}' N^y). \tag{2.14}$$

We rescale and with the notation $S_1(x) = x^{1/y} \tilde{S}_1(1/x)$ we have

$$\gamma(\tilde{C}_2 \tilde{h})^{1/y} \tilde{S}_1(1/\tilde{C}_2 \tilde{h} N^y) = \gamma'(\tilde{C}_2' \tilde{h}')^{1/y} \tilde{S}_1(1/\tilde{C}_2' \tilde{h}' N^y). \tag{2.15}$$

However, in the limit $\mu = hN^y \to \infty$, $\tilde{S}_1(0)$ is a universal number, independent of γ and γ' and also independent of the normalization of ϕ. Thus, defining the metric constant C_2 with respect to the field h which couples to the correctly normalized field ϕ, rather than $\tilde{h}$, we have for the ratio ρ of the two metric constants

$$\rho = (\gamma/\gamma')^{-y}. \tag{2.16}$$

How does this work out numerically? Since the A_4 RSOS model is by construction already completely normalized [27], from (2.12) and (2.16) we have that the Ising metamagnet Hamiltonian (1.2) should be normalized by a factor $\gamma^{-1} = 4.36 \pm 0.04$ which agrees well with our previous finding $\zeta = 4.3 \pm 0.1$. A second check is provided by the 2D Ising model with an anisotropy in spin space and perturbed with the energy density. In that model the exactly known scaling functions only depend on the variable z/η ($z = (t_1 - 1)N$, $y = 1$), in agreement with $\zeta \sim \eta$, where η parametrizes the anisotropy [40]. Besides looking at the scaling functions directly, this argument provides an additional test of universality.

Our result can be rephrased to the statement that $C_2\gamma^y$ is universal. Although we restricted the discussion to the order parameter perturbation, the argument can be repeated for any relevant scaling field. This implies that knowing one of the metric constants fixes all the others, up to normalization of the scaling fields.

In principle, one could also change the definition of h by an arbitrary factor, that is, one could work with $\hat{h} = \alpha\tilde{h} = \alpha\gamma h$ instead. The influence of α can be separated from that of the overall normalization of the Hamiltonian, however, since the ratio of the masses m_1'/m_1 in the $\mu \to \infty$ limit depends by the above argument leading to (2.16) only on the ratio α'/α of the two realizations since $\tilde{S}_1(0)$ does not depend on the normalization of ϕ. As an example, consider again figure 4. Note that the Ising metamagnet data were obtained from the un-normalized Hamiltonian (1.2), but we have already seen that this is not important in the $\mu \to \infty$ limit. Since in the $\mu \to \infty$ limit, the scaling functions are identical, we have that $\alpha'/\alpha = 1$ and since it is known that the RSOS operator has $\alpha = 1$ [27], we conclude that the staggered magnetic field is equal to the conformal field φ_{22} (up to finite-size corrections).

Of course this discussion does not imply that the non-universal metric factors have disappeared; we have merely traced them back to choosing a proper normalization of the scaling fields.

We now turn to a comparison of the scaling functions between two different universality classes. Since we have already seen that in the $\mu \to \infty$ limit, the mass ratios of the tricritical Ising model are equal to those of the usual Ising model, it is tempting to ask whether there are more similarities. Scaling functions for the usual Ising model were already computed from the un-normalized quantum Hamiltonian [8]

$$H = -\sum_{n=1}^{N} \{t_I\sigma^x(n) + \sigma^z(n)\sigma^z(n+1) + h_I\sigma^z(n)\}. \tag{2.17}$$

The critical point is at $t_I = 1$, $h_I = 0$ and the appropriate scaling variable is

$$\mu_I = h_I N^{15/8}. \tag{2.18}$$

Consider the lowest mass m_1 for both models and define the quantity $L_1(\mu)$ via

$$m_1 = 2\pi x N^{-1} + h^{1/y} L_1(\mu) \tag{2.19}$$

where one has $h = h_I(h_s)$, $x = \frac{1}{8}(\frac{3}{40})$ and $y = \frac{15}{8}(\frac{77}{40})$ for the Ising (tricritical Ising) model, respectively and μ is the corresponding scaling variable. For the tricritical Ising model, L_1 is equal to H_1, but it is not for the Ising model. In figure 5, $L_1(\mu)$ is displayed for the two distinct universality classes. Restricting our attention to values of μ_I larger than $\simeq 2$, we observe that at least for the range of μ considered, these two functions appear to be the same, provided the scales of their arguments are chosen properly. We find

$$\rho' = C_2(\text{Ising})/C_2(\text{RSOS}) = 4.49 \pm 0.09. \tag{2.20}$$

We did not find a similar relationship for the higher masses and even for L_1, the observed relation does not hold for small values of μ.

This observation is stronger than the usual universality between different members of the same universality class. Rather, we have an example for a group of systems which at their critical points $\mu = 0$ are in different universality classes but, when μ is increased, essentially losing their memory on the critical point they started from. This observation yields additional evidence that indeed both the 2D Ising and tricritical Ising models perturbed by φ_{22} (the order parameter) should be described by the same massive particle field theory. In particular, this means that the bulk scaling amplitudes B_i of $\xi_i^{-1} = B_i h^{1/y}$ are the same in the two models, with the normalization of h fixed

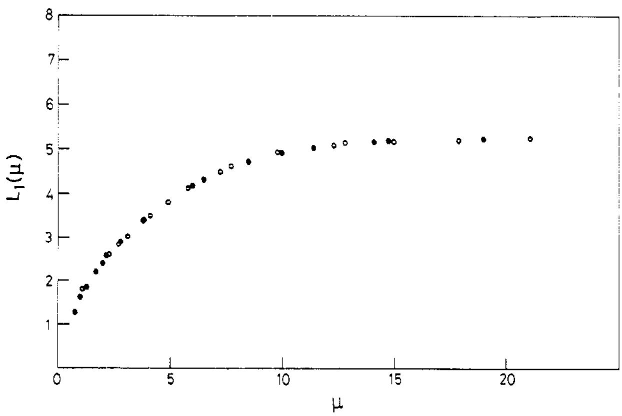

Figure 5. Comparison of the scaling functions $L_1(\mu)$ of the lowest masses for the Ising model (○) and the tricritical Ising model (●). The scaling variable plotted is μ_I and $\rho' = 4.49$.

as discussed above. At this point, we also mention that preliminary calculations for the A_5 RSOS model (corresponding to the tetracritical Ising model) perturbed with φ_{22} also seems to yield mass ratios $r_1 \simeq 1.62$, $r_2 \simeq 1.99$ and it is suggestive to conjecture that this holds for the other A_n RSOS models as well.

To summarize this section, we have given evidence that the tricritical Ising model in an external field should be described by the same theory as the 2D Ising model in a magnetic field (see (2.8) and figure 5). Our results for the scaling functions (figure 4) are in agreement with universality and this serves as an additional check against computational errors.

3. Perturbative calculations and ideas

Having examined the 2D tricritical Ising model numerically, we now try to compare our results with perturbative arguments. First, we shall try to find integrals of motion following [7]. Afterwards, we shall examine a proposal of Dotsenko [32, 41] for the perturbative calculation of scaling functions.

3.1. *Integrals of motion*

We begin by considering the integrals of motion of the perturbed Hamiltonian

$$H = H_c + \lambda \int \mathrm{d}z\, \mathrm{d}\bar{z}\, \varphi(z, \bar{z}). \tag{3.1}$$

Then, using the *assumption* that the space of states of the perturbed theory has the same structure as the one for $\lambda = 0$ (we ignore the possibility of level crossings, see [38]), one can show that the perturbed equations of motion only pick up a finite number of terms and that it is normally sufficient to consider only first-order perturbations (under mild technical conditions, which are always satisfied for order parameter perturbations). The existence of integrals of motion (IM) of the form $T_s(z, \bar{z}) = :T^s(z, \bar{z}):$ can then be demonstrated by a simple counting argument [7, 12]. Define the numbers $\hat{\Lambda}_s$ and $\hat{\Phi}_s$ by

$$\begin{aligned} \sum_{s=0}^{\infty} q^s \hat{\Lambda}_s &= (1-q) q^{c/24} \chi_0(q) + q \\ q^{\Delta} \sum_{s=0}^{\infty} q^s \hat{\Phi}_s &= (1-q) q^{c/24} \chi_\varphi(q) \end{aligned} \tag{3.2}$$

where $\chi_0(q)$ and $\chi_\varphi(q)$ are the Virasoro character functions of the identity operator and the perturbing operator φ of conformal weight Δ in (3.1), respectively and c is the central charge. Now, if one has for some s that $\hat{\Lambda}_{s+1} \geqslant \hat{\Phi}_s + 1$, then there does exist at least one IM of the given form [7].

While this counting argument establishes the existence of non-trivial integrals of motion for the perturbations φ_{12}, φ_{21} and φ_{13} [11, 12], it does not tell us anything on the φ_{22} perturbation. In table 2, we list some values for $\hat{\Lambda}_s$ and $\hat{\Phi}_s$ for odd values of s (since there is no charge conjugation in the tricritical Ising model, there can be no IM with s even). The only recognized conserved quantity off the critical point has spin $s = 1$, which corresponds to energy-momentum conservation.

Table 2. Dimensions of characteristic spaces for the identity operator and the order parameter, corresponding to the conformal fields φ_{11} and φ_{22}.

s	1	3	5	7	9	11	13	15	17	19	21	23
$\hat{\Lambda}_{s+1}$	1	1	2	3	4	6	9	12	18	25	34	49
$\hat{\Phi}_s$	0	1	2	3	5	8	12	19	28	40	59	84

In order to control the possibility that the counting argument might hide an existing IM, we checked the $s=7$ case explicitly, since $s=7$ corresponds to the first non-trivial IM in the 2D Ising model with a magnetic field. By analogy with the procedure for the fields φ_{12}, φ_{21} and φ_{13} [42], one can show that the primary field φ_{22} satisfies the equation

$$\left(L_{-4}-\frac{4\Delta}{9}L_{-2}^2-\frac{4\Delta+15}{6\Delta+18}L_{-1}L_{-3}+\frac{2\Delta+3}{3\Delta+9}L_{-1}^2L_{-2}-\frac{1}{4\Delta+12}L_{-1}^4\right)\varphi_{22}=0 \tag{3.3}$$

where the L_n are generators of the Virasoro algebra and $\Delta=\Delta_{22}$ from the Kac table [1]. Using previous results [43], it is already clear that there will be no IM for neither $s=3$ nor $s=5$. We are looking for a quantity T_8 in the conformal tower of the unity operator, such that $L_0T_8=8T_8$ and satisfying the conservation law

$$\partial_{\bar{z}}T_8=\partial_z Q_6 \tag{3.4}$$

with Q_6 some local operator. As can be read from table 2, there is a three-dimensional space from which T_8 could be constructed. A convenient basis is

$$T_8^{(1)}=L_{-2}^4 I \qquad T_8^{(2)}=L_{-3}^2L_{-2}I \qquad T_8^{(3)}=L_{-4}^2 I \tag{3.5}$$

where I is the unit operator. The effect of $\partial_{\bar{z}}$ can be calculated using known techniques. One can show [6] that there exists a set of operators D_n such that $\partial_{\bar{z}}=\lambda D_0$ and

$$\begin{aligned}&[L_n, D_m]=-((1-\Delta)(n+1)+m)D_{n+m}\\&D_{-n-1}I=\frac{1}{n!}L_{-1}^n\varphi_{22}.\end{aligned} \tag{3.6}$$

Then it follows

$$\begin{aligned}\partial_{\bar{z}}T_8^{(1)}&=-\lambda(\Delta-1)\{4(\Delta^2-5\Delta+9)L_{-7}+12(\Delta-2)L_{-5}L_{-2}+12L_{-3}L_{-2}^2\\&\qquad+12(\Delta-3)L_{-3}L_{-4}\}\varphi_{22}\\\partial_{\bar{z}}T_8^{(2)}&=-\lambda(\Delta-1)\{4(4\Delta+19)L_{-7}-8L_{-3}L_{-4}-12L_{-5}L_{-2}\}\varphi_{22}\\\partial_{\bar{z}}T_8^{(3)}&=-\lambda(\Delta-1)\{60L_{-7}\}\varphi_{22}\end{aligned} \tag{3.7}$$

where we omitted all terms which are just total derivatives $\partial_z(\ldots)$ of local operators. While the term proportional to $L_{-3}L_{-2}^2\varphi_{22}$ can be eliminated by (3.3), there remains a three-dimensional space and a conservation law can only occur for those values of Δ where the coefficient matrix of $\partial_{\bar{z}}$ in the space spanned by (3.7) becomes singular. This leads to the equation

$$4\Delta^3-116\Delta^2+31\Delta+108=0 \tag{3.8}$$

which has the solutions

$$\Delta_1=-0.8303\ldots \qquad \Delta_2=1.1331\ldots \qquad \Delta_3=28.6971\ldots \tag{3.9}$$

which means that there is no unitary model such that a φ_{22} perturbation can be relevant and have at the same time an IM with $s=7$ (for the Ising model, the identity $\varphi_{12}=\varphi_{22}$

provides an additional constraint which guarantees the existence of an IM). Looking at table 2, there do not seem to be too many prospects in finding an IM for yet higher values of s.

Although we did not succeed in finding a non-trivial IM, the numerical results for the mass ratios and the finite-size scaling functions suggest that also a systems perturbed with φ_{22} might be integrable as well, but one probably should look for a more general ansatz in the construction of IMs. If we *knew* that the S-matrix of the tricritical Ising model perturbed by φ_{22} factorized, the two identical mass ratios (2.8) would be enough (see [6]) to establish the identity of the entire S-matrices, while the relationship of the scaling functions (figure 5) would merely fix the ratio of the coupling constants (as can be seen from Lüschers asymptotic expansion for the masses on the a finite lattice [44]). At present, it is an open question why both the Ising and the tricritical Ising models in a symmetry-breaking magnetic field have apparently the same mass spectrum.

3.2. *Scaling functions*

We now turn to a different question and consider the possibility of calculating (reduced) finite-size scaling functions from perturbation theory around the conformal invariant critical point. For the 2D Ising model with a thermal perturbation, scaling functions were calculated successfully by the perturbative approach for the correlation functions in the infinite 2D plane [41] as well as for the correlation length in the strip geometry [45].

Recently, Dotsenko [32] has suggested an extension of his technique [41] to include magnetic perturbations in the Ising model. In order to obtain a scaling function (of the two-point correlation function) which also contains non-analytic terms in the scaling variable as it is required from dimensional counting, he suggests that the perturbed Hamiltonian might pick up an extra term and reads in the case of the Ising model [32]

$$H = H_c + h\sigma + Ah^{\psi}\varepsilon \tag{3.10}$$

because the Ising model order parameter σ produces the energy density ε in the operator product expansion

$$\sigma\sigma \sim 1 + \varepsilon \tag{3.11}$$

where the exponent ψ is fixed by dimensional analysis [32]

$$\psi = \frac{2 - x_\varepsilon}{2 - x_\sigma} = \frac{8}{15} \tag{3.12}$$

for the Ising model and A is a free (probably model-dependent) parameter.

Let us check this idea of a possible dynamical generation of additional terms in the Hamiltonian. To do so, we calculate the finite-size scaling functions G_i of the lowest masses in the Ising model from (3.10) on the strip. The calculation is completely parallel to the one done in the infinite plane [32]. Since the first-order term from σ vanishes and $3\psi < 2$, the lowest orders of G_i are determined by the operator ε. The calculation was already carried out by Reinicke [45], and we obtain, with the normalization of H chosen as in (2.17)

$$m_1 = h_I^{8/15}\left(\frac{\pi}{2}\mu_I^{-8/15} + A + A^2\frac{\ln 2}{\pi}\mu_I^{8/15} + 1.260570\mu_I^{22/15} - A^4\frac{3\zeta(3)}{32\pi^3}\mu_I^{24/15} + \dots\right) \tag{3.13}$$

$$m_2 = h_I^{8/15}\left(4\pi\mu_I^{-8/15} + \frac{2}{\pi}A^2\mu_I^{8/15} + 0.642422\mu_I^{22/15} + \dots\right) \tag{3.14}$$

for the two lowest masses where ζ is the Riemann zeta function. Similarly, for the tricritical Ising model we write in the same spirit

$$H = H_c + h\sigma + Ah^{\psi}\varepsilon + A'h^{\psi'}v \tag{3.15}$$

where A, A' are free parameters and from dimensional analysis as above

$$\psi = \frac{2 - x_\varepsilon}{2 - x_\sigma} = \frac{72}{77} \qquad \psi' = \frac{2 - x_v}{2 - x_\sigma} = \frac{32}{77} \tag{3.16}$$

as motivated by the operator product expansion [42] (irrelevant operators are discarded)

$$\sigma\sigma \sim 1 + \varepsilon + v \tag{3.17}$$

where v is the vacancy operator (corresponds to φ_{13}). Using the known operator product expansion coefficients for the tricritical Ising model [46, 47] we obtain

$$m_1 = \zeta h_s^{40/77}\left(\frac{3\pi}{20}\mu^{-40/77} + 9.403A'\mu^{-8/77} + 1.501A\mu^{32/77} + O(A'^2\mu^{24/77}, \mu^{114/77})\right) \tag{3.18}$$

$$m_2 = \zeta h_s^{40/77}\left(\frac{2\pi}{5}\mu^{-40/77} + O(A'^2\mu^{24/77}, \mu^{114/77})\right) \tag{3.19}$$

where ζ is the normalization factor of the Ising metamagnet. With these expressions in hand, we now compare with the numerical results for the scaling functions. We note the following.

1. The presence of the free parameters A, A', which cannot be absorbed into a renormalization of the scaling variable, is in contradiction to the usual expectation of universality (see equation (2.11)) of the scaling functions [39]. The universality of the finite-size scaling function in the 2D Ising model with a magnetic field (and for the three-states Potts model as well) has been already checked earlier [48] and we did so in section 2 (see figure 4) for the tricritical Ising model. The numerical evidence for universality already suggests that the extra terms proposed can at most make a very small contribution to the scaling functions.

2. For a detailed quantitative check, we consider the scaling functions of the Ising model. To make such a comparison meaningful, the scaling variable must be small enough to guarantee the applicability of the perturbative scheme employed. This will be the case if the correlation length ξ_1 is still large compared with the lattice size and this implies in turn that the mass ratios r_i should be close to their values at $\mu = 0$, known from conformal invariance. This condition is satisfied for the lattices we use here up to about $\mu_I \simeq 0.2$. However, since the reduced scaling functions H_i we are interested in contribute less than 1% to the masses for these values of μ_I, finite-size corrections must be taken into account. We thus obtain the reduced scaling functions not from (2.10), but rather from [31, 40]

$$\begin{aligned} m_1 &= \frac{\pi}{2}N^{-1} + \frac{\pi^3}{96}N^{-3} + h_I^{8/15}H_1(\mu_I) \\ m_2 &= 4\pi N^{-1} - \frac{\pi^3}{6}N^{-3} + h_I^{8/15}H_2(\mu_I) \end{aligned} \tag{3.20}$$

In figure 6, we show the reduced scaling functions H_i for the Ising model. We note that scaling is well satisfied which means that any further corrections to finite-size

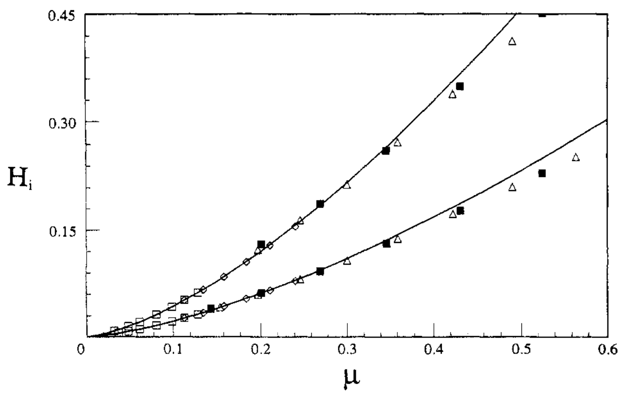

Figure 6. Reduced scaling functions H_1 (upper curve) and H_2 (lower curve) for the Ising model as obtained from (3.30). The full curves are the second-order perturbation predictions (3.13), (3.14) with $A=0$. The symbols denote different values of h_I: □ $3 10^{-4}$–$8 10^{-4}$, ◇ 0.0015, △ 0.004, ■ 0.007.

scaling are still negligible. The full curves are calculated from perturbation theory with $A=0$ (see (3.13, 3.14)), which is in good agreement with the numerical data for $\mu_I \leqslant 0.3$. If μ_I is getting larger, the higher-order terms must be taken into account. We conclude that if the terms proposed by Dotsenko are present at all, the coefficient A must be very small. We find an upper bound for the Ising model of the order $A < 3\times 10^{-4}$. Analogously, for the tricritical Ising model we have already from figure 3 an upper bound $A' < 10^{-2}$. We also recall that Monte Carlo data for the Ising model correlation function do not seem to reproduce the corresponding prediction of Dotsenko either [9].

3. For moderately large values of μ_I (that is about $0.6 \leqslant \mu_I \leqslant 2$) the data are well described by (see figure 4 in [8])

$$m_1 = h_I^{8/15}\left(\frac{\pi}{2}\mu_I^{-8/15} + B\mu_I + \ldots\right) \tag{3.21}$$

with $B \simeq 1.00 \pm 0.05$. Similar results were also observed for the correlation function in the plane [49] but the previous remarks suggest that these simple expressions arise by numerical coincidence.

4. We have seen that for larger values of μ the scaling functions of the Ising model and the tricritical Ising model apparently are the same (see figure 5). It is not clear yet whether one can reproduce this scaling function relationship from conformal perturbation theory. Indeed, the explicit expressions given to lowest orders do not show any sign of similarity.

In summary, our numerical results do not support Dotsenkos' [32] proposal for a perturbative calculation of (finite-size) scaling functions. Rather, we find that conventional perturbation theory is completely sufficient to describe the scaling functions of the correlation lengths and there is no indication for the presence of additional, dynamically generated terms in the Hamiltonian.

4. Conclusions

The two-dimensional tricritical Ising model was studied in an external magnetic field close to the tricritical point. Although this kind of perturbation does not appear to belong to the class of operators which are expected to lead to an integrable off-critical system, numerical calculations have given evidence that this system is described by the same massive field theory as the 2D Ising model in a magnetic field.

The finite-size scaling functions for the inverse correlation lengths were calculated and checked for universality. The form of the scaling functions does not support recent speculations [32] for the dynamical generation of universality-breaking terms of the perturbation series in perturbative calculations starting from the critical point. Rather, the conventional perturbation scheme reproduces well the numerical results of the scaling functions, both for thermal and magnetic perturbations.

Finally, the very fact that a simple picture emerges for the most relevant perturbing operator possible in a minimal model, which is the hardest one to study in a perturbative fashion, indicates the eventual possibility of a considerable generalization of present attempts to explore the entire scaling region of 2D conformally invariant critical points.

Last, but not least, the techniques developed so far lead to the prediction of universal ratios of scaling amplitudes which are directly accessible for experiment.

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